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On the universality class of the conifold

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Abstract: The possibility of having discrete degrees of freedom at singularities associated to "conifolds with discrete torsion" is studied. We find that the field theory of D-brane probes near these singularities is identical to ordinary conifolds, so that there are no additional discrete degrees of freedom located at the singularity. We shed light on how the obstructions to resolving the singularity are global topological issues rather that local obstrucions at the singularity itself. We also analyze the geometric transitions and duality cascades when one has fractional branes at the singularity and compute the moduli space of an arbitrary number of probes in the geometry. We provide some evidence for a conjecture that there are no discrete degrees of freedom localized at any Calabi-Yau singularity that can not be guessed from topological data away from the singularity.

Keywords: Duality in Gauge Field Theories, AdS/CFT Correspondance, Non-Commutative Geometry.

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Contents

It was proposed in [1] that there are conifold singularities in Calabi-Yau manifolds that can not be resolved or deformed. These singularities arise from deforming the complex structure of a $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with discrete torsion by marginal deformations in the string wsorldsheet. As such, it seems that these singularities are new classes of conifolds. For this problem, it is important to understand if this obstruction is a property of the singularity itself, or instead a topological obstruction from the embedding of the singularity in a given Calabi-Yau space. Since the worldsheet conformal field theory is non singular (one can obtain the conifold singularity with an infinitesimal deformation away from the orbifold point) one can use D-branes as probes of the geometry without the fear of them becoming massless at the singularity.

If we consider a large volume Calabi-Yau space (we make the orbifold non compact) we can study the orbifold geometry point with a collection of D3-branes. This gives us a probe of the geometry associated to the orbifold, and produces a near horizon geometry which is dual to a four-dimensional CFT [2, 3, 4]. The rules for

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calculating correlations functions in the field theory and how to compare them to gravity were laid out in [5, 6]. On this orbifold geometry, one can find superpotential deformations of the field theory which correspond to infinitesimal deformations of the complex structure of the Calabi-Yau space [7, 8], and which lead to a conifold singularity in the moduli space of D-branes.

If this "conifold with discrete torsion" is topologically distinct from the standard conifold, then one can compare two distinct field theories that realize the singularity as an infrared fixed point of the RG flow to decide this question. These two field theories are given by the deformed field theory with discrete torsion near the point in the moduli space of vacua where the singularity is located; and the field theory associated to the conifold as realized by Klebanov and Witten [3]. The test will be to see if these two a priori distinct field theories belong to the same universality class of field theories or not.

Orbifold type singularities can be readily understood by the quiver diagram techniques of Douglas and Moore [9]. Experience with these examples has proven that one can in general expect that all of the information of the singularity can be encoded in a four-dimensional conformal field theory, that bulk D-branes fractionate at the singularity and that the moduli space of D-branes will recover the algebraic geometry of the singularity.

The next simplest singularity is the conifold, which has been the subject of a lot of attention through the years, as it presents us with topology changing transitions [10, 11], interacting conformal field theories which do not have a free field theory limit [3], and because it allows us to study confining field theories. These give rise to topology changing transitions [12, 13, 14]. Brane realizations of these theories have been considered in [15, 16, 17].

One needs a systematic approach to construct these field theories and to interpret how brane fractionation occurs. In [18, 19] it was proposed to study the (perturbative) moduli space of these theories in terms of the representation theory of non-commutative algebras. The possibility of fractional branes can then be understood in terms of families of irreducible representations of the algebras which, for certain values of the parameters, become reducible. At these points the D-branes split into smaller irreducible representations. This should be the signal for the presence of singularities. From the knowledge of the algebra and the irreducible representations, one can compute explicitly the quiver diagram of the singularity [19, 20] and obtain results which give exactly the quiver diagrams of Douglas and Moore [9], but which are also applicable to non-orbifold singularities, once the non-commutative algebra is known. Moreover, it has been shown that one can build compact Calabi-Yau spaces which correspond to non-commutative versions of algebraic geometry [21]. Naturally, since we are analyzing singularities in a commutative geometry, one needs a way to understand this commutative structure from the non-commutative point of view. This commutative structure is provided by the center of the algebra.

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Once one understands that fractional branes can arise, one can study how the theories get affected by the presence of fractional branes (if they are allowed by anomaly cancellation), and we should generally expect that these fractional brane constructions lead to field theories that are closely associated to the four-dimensional conformal field theory of the singularity — but which are no longer conformal. These associated field theories are expected to have all kinds of interesting non-perturbative phenomena like confinement and duality cascades [12].

In principle, given a singularity it is unclear whether this collection of associated field theories is unique. There could be more than one algebra[1] which gives rise to the same type of singular geometry. The aim of this paper is to study this problem, namely the question of whether the field theories at singularities in Calabi-Yau threefolds are universal or if there are extra discrete degrees of freedom in string theory which are necessary to specify such a class of theories. Most of our work will be done for the conifold singularity and we will find that, at least in the example of "conifolds with discrete torsion", they are described by the same universality class of the conifold without discrete torsion, but with some relations between the gauge couplings fixed from boundary conditions in the UV.

Towards the end of the paper we give a holographic argument which supports the idea that the quantum field theories of D-branes on any Calabi-Yau singularity are indeed universal (up to gauge field theory dualities [22]). The singularities are allowed to have discrete torsion degrees of freedom only if the singularity is not isolated. This extra data is encoded by the monodromies of fractional branes of the codimension two singularities around the codimension three singularities. As such, these monodromies correspond to holographic data away from the singularity and are specifying the 'stringy' topological type of the singularity. However, our point of view will be that this data is not concentrated at the singularity itself, since it can be measured far away from the singularity. Given this additional data, this should determine completely the universality class of field theories at the singularity (both with and without fractional branes) and the duality cascades between different realizations of the field theory. These field theories associated to the singularity are only given by the infrared fixed point of a collection of D-branes very near the singularity, and in the presence of fractional branes these may produce topology changing transitions and confinement.

The paper is organized as follows:

Section 1 describes the conifold algebra in detail using the techniques of [18, 19] and the moduli space of vacua is shown to be the symmetric product of the conifold geometry.

Section 2 presents a field theory which corresponds to the orbifold with discrete torsion, and a deformation which leads to a conifold singularity. We expand the field

[1] Properly we should talk of equivalence classes of algebras.

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theory about a background of D-branes near the singularity and we show that the IR fixed point of the theory gives rise to the same field theory as the one studied in [3], which corresponds to the conifold without discrete torsion.

Section 3 deals with the reasons why certain marginal deformations of the conifold geometry can be obstructed in the field theory with discrete torsion. We find a consistent picture from various points of view which solve the puzzle presented by [1].

In section 4 we further advance the geometric picture of geometric transitions and the understanding of Seiberg dualities. In particular, we are able to compute the moduli space of an arbitrary number of D3-brane probes in the presence of fractional branes including the non-perturbative superpotential, and we find that it is indeed given by the symmetric product of the deformed conifold.

In section 5 we speculate on universality for more general singularities and propose a holographic argument that gives evidence for any Calabi-Yau singularity to behave universally.

The paper is concluded in section 6.

1. D-branes and the conifold algebra

The conifold geometry can be described by the variety which is the solution to the equation

$$a^2 + b^2 + c^2 + d^2 = 0 \tag{1.1}$$

in four variables, and represents a conical singularity which is also a Calabi-Yau manifold.

As the geometry of the singularity is a cone, one also expects that setting a collection of D-branes at the singularity will produce a conformal field theory [3, 4]. The cone is over the homogeneous space $(\mathrm{SU}(2) \times \mathrm{SU}(2)) / \mathrm{U}(1)$. The $\mathrm{U}(1)$ is embedded diagonally. The group manifold has an $\mathrm{SU}(2)^4$ isometry group which is broken to $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ upon taking the quotient. In the supergravity theory the $\mathrm{U}(1)$ direction will be interpreted as the R-charge of the conformal symmetry group.

A D-brane probe on the geometry should have the conifold as a moduli space of vacua. In general, one can think of having n D-brane probes of the geometry, and it is expected that the classical moduli space of such probes should give us the nth symmetric product of the conifold geometry.

A change of variables lets us write the conifold as

$$uv = wz \tag{1.2}$$

so that the conifold geometry can also be constructed as the holomorphic quotient $\mathbb{C}^4/\mathbb{C}^*$ with charges $[-1, -1, 1, 1]$ on variables a_1, a_2, b_1, b_2. The invariants under such an action are the four $a_i b_j$, which are identified with u, v, w, z.

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We can also rewrite the conifold geometry as

$$uv = (w' - \lambda\gamma)(w' + \lambda\gamma) \tag{1.3}$$

with a parameter λ. We can set later $\lambda = 1$ so that we have a family of conifold singularities, which at $\lambda = 0$ gives us the orbifold space $(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$. Thus, one can obtain the conifold geometry as a deformation of the complex structure of the orbifold $(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$.

For D-branes, this means that one should be able to obtain the field theory of D-branes on a conifold geometry from a superpotential deformation of the supersymmetric field theory associated to the above orbifold, and then take the infrared limit near the conifold singularity in moduli space.

The approach of Klebanov and Witten [3] to the conifold field theory is given by a holomorphic quotient $(\mathbb{C}^4)/\mathbb{C}^*$. The theory has four superfields x_1, x_2, y_1, y_2 with charges $(1, 1, -1, -1)$ under a U(1) gauge field, and with no superpotential. There is obviously an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ symmetry which acts on the x_i and y_i as $(2, 1)$ and $(1, 2)$, respectively.

The moduli space is given by the vevs of the gauge invariant superfields

$$N_{ij} = x_i y_j \tag{1.4}$$

and one finds the classical constraint in terms of the 2×2 matrices $\det(N_{ij}) = 0$, which gives rise to the conifold geometry.

To make further contact with D-branes, one has to interpret how the field theory is affected when one adds N-branes. Clearly one has to extend the gauge group to $\mathrm{U}(N)$ somehow and one wants to be able to break $\mathrm{U}(N) \to \mathrm{U}(1)^N$ by classical vevs of x_i, y_i. If the x_i are in the N and the y_i are in the $\bar{N}$ of $\mathrm{U}(N)$, this means that we need N of the x_i and y_i in order to get N distinct points on the conifold. This would lead to an additional $\mathrm{U}(N)$ symmetry of the theory which is not visible in the supergravity, so this symmetry must be gauged. The $\mathrm{U}(N) \times \mathrm{U}(N)$ is expected from D-brane gauge groups if there are two types of fractional branes. One linear combination of the diagonal $\mathrm{U}(1)'s$ is decoupled (no field is charged under it), while under the other U(1) the fields x, y have opposite charges.

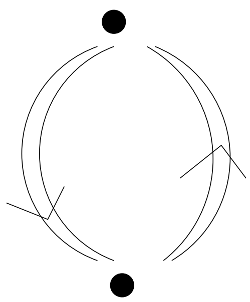

Figure 1: Quiver diagram for the conifold field theory

The end result is a field theory which can be represented by the quiver diagram drawn in figure 1

One also needs to add a superpotential so that the moduli space of vacua of the theory is of dimension N rather than of order N^2 (the number of free fields that make the theory). If one requires the additional symmetry under the exchange of the two gauge groups then one can argue that the fields x_i and y_i have the same dimension

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which has to be set equal to $\dim(x_i) = 3/4$. The R charges of the fields are fixed by the requirement of conformal invariance. The necessary superpotential deformation is quartic and unique given the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ global symmetry and being of single trace type

$$W = \lambda\,\mathrm{tr}(N_{ij} N_{lk} \epsilon_{il} \epsilon_{jk}) \tag{1.5}$$

with $N_{ik} = x_i y_k$. This theory can be shown to be conformal by using the Leigh-Strassler techniques [23], as the β functions for the gauge couplings and the superpotential are related.

To make contact with the algebraic techniques of [18], we can also write this superpotential in terms of fields X, Y which are given by the $2N \times 2N$ matrices

$$\begin{aligned} X &= \begin{pmatrix} 0 & x_1 \\ y_1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & x_2 \\ y_2 & 0 \end{pmatrix}, \\ \sigma &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{1.6}$$

so that

$$W = \lambda\,\mathrm{tr}\left(\sigma[X,Y]^2\right). \tag{1.7}$$

The reason for introducing the discrete variable σ is as a placeholder for the U(1) gauge group in the $\mathrm{U}(N) \times \mathrm{U}(N)$ theory. In the case of one-brane, it is exactly gauging by $\exp(\theta\sigma)$ which gives rise to the correct U(1) action on X, Y, and it is easy to check that the above superpotential gives zero. In terms of the $2N \times 2N$ matrices the $\mathrm{U}(N) \times \mathrm{U}(N)$ gauge group is selected by requiring that it commute with σ.

We also note that

$$X\sigma = -\sigma X\,, \qquad Y\sigma = -\sigma Y\,, \qquad \sigma^2 = 1\,. \tag{1.8}$$

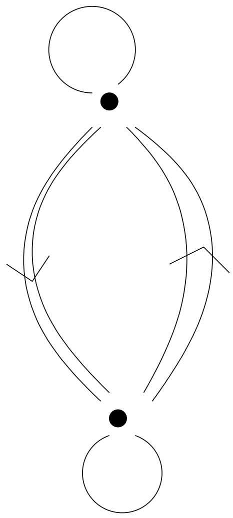

Figure 2: Quiver diagram for the A_1 singularity

There are also constraints derived from the superpotential which read

$$(\sigma[X,Y],X) = (\sigma[X,Y],Y) = 0\,. \tag{1.9}$$

We will call the polynomial algebra generated by X, Y, σ — subject to the relations (1.8) and (1.9) — the algebra of the conifold.

Let us make one last note as to how we can obtain this field theory from the deformation of complex structure of the $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ algebra. From eq. (1.3) we see that we get a family of resolved ADE singularities, with a deformation that depends on the third coordinate γ. These deformations of complex structure are given by twisted fields in the worldsheet conformal field theory, and this translates into F

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terms for the adjoints in the quiver of the $\mathbb{C}^2/\mathbb{Z}_2$ $\mathcal{N}=2$ quantum field theory [24], all we need now is to make them dependent on the third coordinate. The $\mathcal{N}=2$ A_1 theory is described by the quiver diagram in figure 2.

The associated non-commutative algebra is given by the crossed product algebra of $\mathbb{C}^3$ and the group algebra $\mathbb{Z}_2$ [19]. Let σ be the $\mathbb{Z}_2$ group variable, thus $\sigma^2=1$

$$\sigma X = -X\sigma\,, \qquad \sigma Y = -Y\sigma\,, \qquad \sigma Z = Z\sigma\,. \tag{1.10}$$

The superpotential is given by $\mathrm{tr}([X,Y],Z)$, the same as the $\mathcal{N}=4$ field theory, which leads to X,Y,Z commuting with each other. All we have done is added the extra discrete variable σ, which projects us onto the $\mathbb{Z}_2$ invariant states.

Now we need to perturb the theory by a twisted chiral field. These are of the form[2]

$$-\frac{1}{n}\,\mathrm{tr}(\sigma Z^n)\,. \tag{1.11}$$

They only modify the fact that X,Y commute, so we get to

$$[X,Y] = \sigma Z^{n-1}\,. \tag{1.12}$$

From here, the center of the algebra is always given by $Z, u = X^2, v = Y^2, \gamma = XY+YX$, and the relation between these commutative variables changes because when we evaluate γ^2 we need to commute X,Y to order them. In this way,

$$\gamma^2 = 4u^2 + Z^{2(n-1)}\,. \tag{1.13}$$

We need $n=2$ to obtain the conifold. In this case Z has a mass term, so it can be integrated out, and the only surviving generators are X,Y with an induced superpotential given by

$$\mathrm{tr}\left(\sigma([X,Y])^2\right) \tag{1.14}$$

which again reproduces eq. (1.7). As the field Z is integrated out, it dissapears from the quiver diagram and we are left with figure 1 instead.

1.1 On the fate of the U(1)'s

So far we have described the field theory obtained in the infrared as the fixed point of the RG flow from a $\mathrm{U}(N)\times\mathrm{U}(N)$ field theory in the UV which is matched to a string background.

In particular, one describes the UV boundary conditions of the flow by the gauge couplings of the $\mathrm{U}(N)^2$ theory. There is one U(1) which, as far as the low energy field theory degrees of freedom are concerned, is decoupled from the open string sector because no field is charged under it. This field participates in the gauge variation of

[2]This is a rewriting of the results of [25] in these matrix variables.

the closed string B-field and can be gauged away. If the closed string fields decouple from the low energy degrees of freedom then this global U(1) is not part of the low energy dynamics on the brane at all.

There is a second U(1) term in the lagrangian which is the relative U(1) (the difference of the two diagonal U(1) charges in the $\mathrm{U}(N) \times \mathrm{U}(N)$ theory). This term allows us to give a D-term to the lagrangian, and there is matter coupled to this U(1). At the string scale (or some other higher energy scale where we are matching the field theory), this gauge coupling is determined by the two $\mathrm{SU}(N)$ couplings, but as we flow towards the infrared the running gauge coupling goes to zero from screening by the massless charged fields. At the infrared fixed point theory the U(1) is decoupled. When we describe the infrared field theory by an AdS background, this flow towards the infrared has already been taken into account, so in the conformal field theory the U(1) is not properly there any more. If we are on a resolved conifold, the D-term of this U(1) is part of the low energy data, and one has to keep a scaling limit where the coupling of the U(1) goes to zero, but where the effects of the D-term give a finite scale. However, these massless fields are not in the spectrum of the conformal field theory associated to the AdS background.

1.2 Representations of the conifold algebra

Now we want to understand the classical moduli spaces of vacua of the conifold theory. We follow the techniques presented in [18, 19].

From the algebraic relations we find that

$$u = X^2\,, \qquad v = Y^2\,, \qquad w = (XY + YX)\,, \qquad z = [XY - YX]\sigma \tag{1.15}$$

all belong to the center of the algebra, and thus they describe a commutative geometry. One can easily show that

$$4uv = w^2 - z^2 \tag{1.16}$$

which, up to some trivial rescalings, gives us the commutative conifold geometry. On each irreducible representation of the algebra we have that all of u, v, w, z are proportional to the identity, and thus specify a single point in the conifold. It is a simple exercise to show that, away from the singularity, there is one unique irreducible representation of the algebra in terms of 2×2 matrices for each of the conifold points, and they are constructed by giving arbitrary values to the variables x_i, y_i in eqs. (1.6), subject to the gauge equivalence generated by $\exp \theta\sigma$ (which changes the values of the parameters). Each of these representations is such that $\mathrm{tr}(\sigma) = 0$, which means that it is built out of two fractional branes, one of each type.

At the conifold singularity, we find additional irreducible representations of the algebra which are one dimensional and which are given by taking $x_i = y_i = 0$, as all the equations are satisfied automatically except $\sigma^2 = 1$. The two possible solutions

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correspond to $\sigma = \pm 1$, which we will call R_- and R_+. If we have NR_-- and MR_+-branes, then the gauge group is $\mathrm{U}(N) \times \mathrm{U}(M)$.

It is easy to see that the bulk representation decomposes as $R_{\text{bulk}} \to R_- \oplus R_+$. These are the fractional branes. They are characterized by the distinct values of $\mathrm{tr}(\sigma) = \pm 1$, so we can use $\mathrm{tr}(\sigma)$ as the parameter that counts how many fractional branes we have.

The end result is that branes only fractionate at the singularity. There are two types of fractional branes and a bulk brane fractionates into one of each of the two types of fractional branes. The gauge group of a configuration of N bulk branes with $N - M$ fractional branes is the gauge group associated to N and M fractional branes of different kind, namely $\mathrm{U}(N) \times \mathrm{U}(M)$.

At least in principle, the moduli space of the theory is given by a direct sum of irreducible representations of the algebra, and as such it is identified with the a set of N points in the conifold.

1.3 The resolved conifold

Although we have written the holomorphic quotient naïvely as $\mathbb{C}^4/\mathbb{C}^*$ one would really want to be more careful and write the quotient as a symplectic quotient to take into account the metric aspects of the singularity.

If we do this, one needs to consider also the D-term equations of motion of the supersymmetric field theory.

In the conifold field theory language, the D-term constraints are given by

$$\left[\bar{X}, X\right] + \left[\bar{Y}, Y\right] = D\sigma \tag{1.17}$$

or equivalently

$$|x_i|^2 - |y_i|^2 = D\,, \tag{1.18}$$

where we have introduced complex conjugate coordinates for X, Y (which are their adjoints in the matrix algebra and give the algebra the structure of a $\mathbb{C}^*$ algebra). Since we have that $X\bar{X}$ is a gauge invariant function in $\mathbb{C}^4$, it descends to an element of the center in the quotient algebra, and similarly for $Y\bar{Y}$. On the moduli space of vacua eqs. (1.17) are block diagonal for the individual branes. Thus we only need to analyze one bulk brane at a time. All of the individual D-terms will be identical to one another as the supersymmetric field theory can only accommodate non-zero D-terms for the non-decoupled $\mathrm{U}(1)$ in the $\mathrm{SU}(N) \times \mathrm{U}(1) \times \mathrm{SU}(N) \times \mathrm{U}(1)$ field theory.

We already have representations of X, Y in terms of 2×2 matrices as described in eqs. (1.6), and we take $\bar{X}, \bar{Y}$ to be the adjoints in this representation. The conifold is described by the level set $D = 0$. In this case, the representations at the origin are those for which $x_i y_j = 0$ so if we assume one of the $x_i \neq 0$ then both of the y_j are zero and from (1.17) we find that $|x|^2 = 0$ as well. In this case the holomorphic branes truly fractionate at the origin.

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For distinct level sets, e.g. $D > 0$, we have that $|x|^2 > 0$, so the origin of the conifold in moduli space has a non-zero value for (at least one of) the coordinates x_1, x_2. The ratio

$$\kappa_{ij} = \frac{x_i}{x_j} = \frac{N_{ij}}{N_{jj}} \tag{1.19}$$

is a holomorphic invariant for this configuration so long as the denominator is non zero. This requires us to split the coordinates in moduli space in two patches where $x_{1,2} \neq 0$ and $y = 0$.

As such, when we include these two patches we have a $\mathbb{CP}^1$ space, as $\kappa_{12}\kappa_{21} = 1$ when both are defined. It is also clear that the interpretation of the origin in moduli space is that one has blown-up the singularity to a finite size $\mathbb{CP}^1$ as follows from the description in terms of quotients of commutative variables in eq. (1.19).

Choosing $D < 0$ exchanges the role of x, y and corresponds to flopping the rigid $\mathbb{CP}^1$.

From the holomorphic quotient point of view, these two blow-ups can be interpreted as holomorphic quotients

$$\frac{(\mathbb{C}^4/E)}{\mathbb{C}^*}, \tag{1.20}$$

where we remove one of two exceptional sets: either the complex variety $y_i = 0$ or the complex variety $x_i = 0$ so that irreducible representations with these values of the complex parameters are not allowed. Presented in this manner, there is no need to specify values for the D-terms and one is only using algebraic geometric descriptions of the blow-ups.

2. The conifold with discrete torsion

One can also obtain a conifold singularity from a deformation of the complex structure of the $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with discrete torsion [1].

A non-compact version of this space is a $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ with discrete torsion. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold is given by the variety described by the equations

$$\gamma^2 = uvw \tag{2.1}$$

in four variables. A deformation that takes us to a conifold singularity is given by

$$\gamma^2 = uvw - \lambda^2(u + v + w) + 2\lambda^3 \, . \tag{2.2}$$

The conifold singularity is located at $u = v = w = \lambda$. This deformation can be realized by a quantum field theory, so we can engineer the above geometry. The field theory that realizes the above deformation is associated to a collection of D3-branes at the orbifold with discrete torsion $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$.

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In the case with discrete torsion, the quantum field theory for the orbifold $\mathbb{Z}_2 \times \mathbb{Z}_2$ singularity consists of a quiver diagram with one gauge group and three superfields X, Y, Z in the adjoint representation [7]. This is because the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group has only one irreducible projective representation.The superpotential is given by

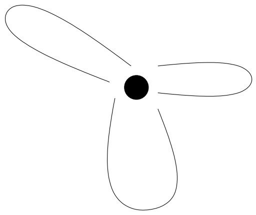

Figure 3: Quiver diagram for the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with discrete torsion

$$a\,\mathrm{tr}(XYZ + ZYX)\,, \tag{2.3}$$

where a is a coupling constant that we set equal to one by rescaling the fields. The field content is summarized by the quiver diagram appearing in figure 3.

This field theory was analyzed from the AdS/CFT correspondence in [7, 18] and shown to lead to the correct moduli space of D-branes plus the fractional branes at the codimension two singularities.

The superpotential deformation that gives rise to the conifold singularity is obtained by adding F-terms for the superfields [7, 8]

$$-2\zeta\,\mathrm{tr}(X + Y + Z) \tag{2.4}$$

and it is chosen so that one has a S_3 symmetry from permuting the three variables X, Y, Z.

We want to show now that this superpotential leads to the variety (2.2). The modified algebra is as follows

$$\begin{aligned} XY + YX &= 2\zeta\,, \\ XZ + ZX &= 2\zeta\,, \\ YZ + ZY &= 2\zeta\,. \end{aligned} \tag{2.5}$$

It is easy to verify that $u = X^2$, $v = Y^2$, $w = Z^2$ are in the center of the algebra for any value of ζ. At $\zeta = 0$ also $\gamma = XYZ$ is in the center, but this gets modified to

$$\gamma = XYZ - \zeta X - \zeta Z + \zeta Y = -ZYX + \zeta X + \zeta Z - \zeta Y\,. \tag{2.6}$$

One can now expand γ^2 by using the commutation relations and one finds

$$\gamma^2 = XYZXYZ + \{XYZ, -\zeta X - \zeta Z + \zeta Y\} + \zeta^2\left(X^2 + Y^2 + Z^2\right) - 2\zeta^3\,. \tag{2.7}$$

From eq. (2.6), we replace $XYZ = -ZYX + 2\zeta X + 2\zeta Z - \zeta Y$ so the relation becomes

$$\begin{aligned} \gamma^2 &= -uvw + (2\zeta X + 2\zeta Z - 2\zeta Y)XYZ + \\ &\quad + \{XYZ, -\zeta X - \zeta Z + \zeta Y\} + \zeta^2\left(X^2 + Y^2 + Z^2\right) - 2\zeta^3 \end{aligned} \tag{2.8}$$

which can be written in terms of a commutator

$$\gamma^2 = -uvw + [(\zeta X + \zeta Z - \zeta Y), XYZ] + \zeta^2\left(X^2 + Y^2 + Z^2\right) - 2\zeta^3 \tag{2.9}$$

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but since γ is central, we obtain from (2.6) that the commutator in the above expression vanishes and we are left with

$$\gamma^2 = -uvw + \zeta^2(u+v+w) - 2\zeta^3 \tag{2.10}$$

which is identical to eq. (2.2).

For simplicity, we will now set $\zeta = 1$.

The above algebra has one two-dimensional representation for each point away from the singularity. Let us assume that $X^2 = a^2 \neq 0$. Then, since X^2 is in the center while X is not, we find that

$$X = a\sigma_3 \,. \tag{2.11}$$

From here it follows that

$$Y = a^{-1}\sigma_3 + b\sigma_2 + c\sigma_3 \,. \tag{2.12}$$

With $Y^2 = a^{-2} + b^2 + c^2$. We can generically make the choice $b \neq 0$ if $Y^2 \neq 1/X^2$. Finally, we can take

$$Z = a^{-1}\sigma_3 + \left(1 - a^{-2}\right) b^{-1}\sigma_2 \,. \tag{2.13}$$

We can now first take the limit $a \to 1$ and then take $b, c \to 0$, which shows that all X, Y, Z become identical to each other. Thus, they commute and the representation becomes reducible into two one-dimensional irreducible representations which are given by $X = Y = Z = \pm 1$. Notice that the representation becomes reducible exactly at the conifold singularity when we evaluate explicitly γ, u, v, w.

It is an easy matter to establish that for these fractional brane configurations the D-term equations are satisfied and hence the D-branes fractionate at the geometric singularity. The representation splits into two distinct irreducibles and this is our first evidence that the two singularities are describing the same universality class of four-dimensional field theories, as both have the same splitting of fractional branes at the singularity.

2.1 Field theory for the conifold with discrete torsion

Now we want to prove that the infrared dynamics of D-branes near the conifold singularity (this is a point in the moduli space of vacua of the theory) gives exactly the same low energy effective field theory as the conifold without discrete torsion.

To do this, we need to go to the region of moduli space where the branes fractionate and expand the theory about that background. Since we have a $\mathbb{Z}_3$ symmetry that exchanges $X \to Y \to Z$, it is convenient to do a linear change variables so that the new variables have definite charges under this $\mathbb{Z}_3$ symmetry. These will be called A, B, C, and we write

$$\begin{aligned} X &= A + B + C \,, \\ Y &= A + \eta B + \eta^2 C \,, \\ Z &= A + \eta^2 B + \eta C \,, \end{aligned} \tag{2.14}$$

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Where η is a cube root of unity. Under the $\mathbb{Z}_3$ symmetry, A is neutral while B, C have opposite charges. The superpotential in these variables is given by

$$W = \mathrm{tr}\left[2\left(A^3 + B^3 + C^3\right) - 3(ABC + CBA) - 6A\right]. \tag{2.15}$$

And the two conifold singularity irreducible representations corresponds to $B = C = 0$ and $A = \pm 1$. We now want to take N D-branes close to the conifold, which gives us a $\mathrm{U}(2N)$ theory that we expand in fluctuations about the conifold point in the moduli space of vacua. Since $A = \pm 1$, the gauge group is broken to $\mathrm{U}(N) \times \mathrm{U}(N)$ and one can draw a naïve quiver from decomposing the field content under the unbroken gauge group quantum numbers. This is depicted in figure 4.

The expansion in fluctuations is given by

$$A = \begin{pmatrix} 1 + \delta A_{11} & 0 \\ 0 & -1 + \delta A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} \delta B_{11} & \delta B_{12} \\ \delta B_{21} & \delta B_{22} \end{pmatrix},$$
$$C = \begin{pmatrix} \delta C_{11} & \delta C_{12} \\ \delta C_{21} & \delta C_{22} \end{pmatrix}, \tag{2.16}$$

where we have used the allowed gauge invariance to set $\delta A_{12} = \delta A_{21} = 0$. The equations to first order read

$$\{\langle A\rangle, \delta B\} = \{\langle A\rangle, \delta C\} = \{\langle A\rangle, A\} = 0 \tag{2.17}$$

so we find that B_{ii}, C_{ii}, A_{ii} have to be zero, this is, they are massive fields that have to be integrated out. The massless fields are given by the elements of the B_{ij} and C_{ij} matrices with $i \neq j$. As such, we remove them from the naïve quiver diagram, as well as the arrows corresponding to $\delta A_{12}, \delta A_{21}$ which are the goldstone modes that are eaten by the Higgs mechanism.

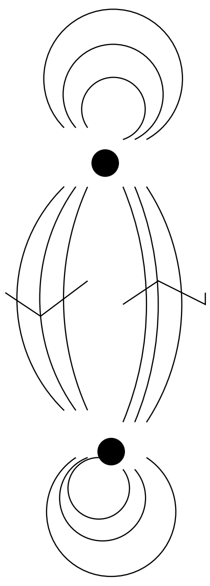

Figure 4: Naïve quiver diagram for the conifold with discrete torsion

The associated quiver diagram of the singularity is then given by two $\mathrm{U}(N)$ gauge groups plus two fields in the $(\bar{N}, N)$ and two fields in the $(N, \bar{N})$, and this begins to look like the standard conifold [3], as described by figure 1. To find the effective superpotential, we have to integrate out the massive fields A_{ii}, B_{ii}, C_{ii}. To do this, it is necessary to expand the equations to second order in fluctuations. From here, we find that

$$\begin{pmatrix} \delta B_{11}^{(2)} & 0 \\ 0 & -\delta B_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} \delta C_{12}\delta C_{21} & 0 \\ 0 & \delta C_{21}\delta C_{12} \end{pmatrix},$$
$$\begin{pmatrix} \delta C_{11}^{(2)} & 0 \\ 0 & -\delta C_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} \delta B_{12}\delta B_{21} & 0 \\ 0 & \delta B_{21}\delta B_{12} \end{pmatrix},$$
$$4\begin{pmatrix} \delta A_{11}^{(2)} & 0 \\ 0 & -\delta A_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} \delta B_{12}\delta C_{21} + \delta C_{12}\delta B_{21} & 0 \\ 0 & \delta B_{21}\delta C_{12} + \delta C_{21}\delta B_{12} \end{pmatrix}. \tag{2.18}$$

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We now have to substitute these terms in the superpotential to obtain the effective superpotential for the massless fields. We get a superpotential of the form

$$W = \mathrm{tr}\langle A\rangle \left(a\delta B_{(1)}^2 \delta C_{(1)}^2 + b\delta B_{(1)}\delta C_{(1)}\delta B_{(1)}\delta C_{(1)}\right) + \text{higher order}\,. \tag{2.19}$$

Although at first sight it looks like the superpotential is different from (1.7), we have to remember that this is only well defined up to field redefinitions. Indeed, for the number of branes set equal to one, $(N = 1)$ we find no superpotential at all, which is exactly the case for the conifold without discrete torsion as well.

Consider now the following commutator

$$\begin{aligned} \zeta &= \left[\begin{pmatrix} 0 & \delta B_{12} \\ \delta B_{21} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\lambda\delta C_{12} \\ \lambda'\delta C_{21} & 0 \end{pmatrix}\right] = [B, C'] \\ &= \begin{pmatrix} \lambda'\delta B_{12}\delta C_{21} + \lambda\delta C_{12}\delta B_{21} & 0 \\ 0 & -(\lambda'\delta C_{21}\delta B_{12} + \lambda\delta B_{21}\delta C_{12}) \end{pmatrix}, \end{aligned} \tag{2.20}$$

where we have defined a modified C' with λ, λ' unitary. It follows that

$$\theta\,\mathrm{tr}\langle A\rangle\zeta^2 = \theta\,\mathrm{tr}\left[\langle A\rangle\left(2\lambda\lambda'\left(\delta B_{(1)}^2\delta C_{(1)}^2\right) + \left(\lambda^2 + \lambda'^2\right)\left(\delta B_{(1)}\delta C_{(1)}\right)^2\right)\right]. \tag{2.21}$$

So we can adjust the coefficients $\lambda, \lambda', \theta$ to match a, b. With this slight change of variables we see that we are getting the right form of the superpotential as in eq. (1.7) with an identification of the variables.

The higher order terms in the superpotential are present but become irrelevant in the low energy field theory once we flow to the infrared fixed point at the conifold singularity. The dictionary between the new variables and the previous ones[3] is $\langle A\rangle \sim \sigma$, $B \sim \tilde{X}$, $C' \sim \tilde{Y}$

Also notice that the $\mathbb{Z}_3$ symmetry under which B, C transformed becomes part of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ expected symmetry of the field theory in the infrared, so there are no additional discrete symmetries in the theory with discrete torsion.

The higher order terms which are irrelevant in the infrared and which we discarded when going to the infrared fixed point break the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ explicitly. Thus, the symmetry of the theory in the infrared is an enhanced global symmetry which exists only in the infrared fixed point field theory.

3. Marginal deformations

We have obtained through very different means the gauge theory corresponding to the conifold, and as such we should be able to match the marginal deformations from

[3] We denote the variables for (1.7) with a tilde.

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the two theories in the infrared fixed point. In particular, the gauge theory at the origin has two marginal deformations corresponding to the two gauge couplings in the quiver diagram, while the superpotential coupling remains fixed from the conditions for conformal invariance.

In the theory obtained from the $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ deformation, we clearly have two distinct gauge couplings in the UV field theory which are also marginal. These are the two gauge couplings in the A_1 quiver theory. Thus, apart from some flow of couplings to the infrared region, we can identify the marginal deformations in the IR with marginal deformations in the UV finite field theory.

The operator is in this case

$$\int d^2\theta \, \mathrm{tr}\left(\sigma W^2\right) \tag{3.1}$$

with $\sigma W = W\sigma$ from the orbifold projection into the physical fields.

For the second case obtained from the orbifold with discrete torsion, there is only one gauge coupling in the UV finite field theory because we have only one gauge group.

However, the marginal operator in the infrared field theory should correspond to some operator in the UV. Since the operator should affect the gauge couplings as they split at the representation, it is proportional to σ. Remember now that we had identified $\sigma \sim \langle A \rangle$, and that the gauge coupling is a holomorphic function of the chiral fields, so the smallest dimension operator in the UV field theory which can accomplish this splitting is

$$\mathcal{O} = \int d^2\theta \, \mathrm{tr}\left(AW^2\right) \tag{3.2}$$

which is of dimension five. Thus, in the UV field theory, this would be considered an irrelevant coupling (in terms of the flow to the infrared) and would be discarded. However, in the flow towards the infrared field theory, this operator acquires an anomalous dimension that makes it relevant at the IR fixed point, making it another example of a dangerously irrelevant operator [23].

However, in the AdS/CFT correspondence one would not add such an operator to the theory in the UV because it would drastically change the boundary conditions.

Thus, in this case, the theory flows to the conifold field theory at some specific value of the couplings. Classically the two coupling are equal and since the equations of motion have a symmetry under $A \to -A$ this means that we are in the scenario where the gauge theory has an extra $\mathbb{Z}_2$ symmetry at the conifold, so this should be true quantum mechanically as well.

For the D-terms of the theory the same type of argument works. We need to add terms to the effective action proportional to σ. But these, not being holomorphic, require us to think of terms in the Kähler potential.

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However there are no terms in the Kähler potential that can generate these types of D terms. From the superfield formulation, terms linear in the D-terms of the vector field will be found in

$$\int d^4\theta \,\mathrm{tr}\left(f\left(\phi, e^{-}V\bar{\phi}e^{V}\right)\right) \tag{3.3}$$

and to first order this involves the commutators $[\bar{\phi}, V]$. But these are exactly the gauge fields which are integrated out by the Higgs mechanism. These deformations, not being described by a superspace integral, necessarily break supersymmetry.

The obstruction to being able to blow up the conifold tip has to do with the fact that if such a blow-up were possible, then the 2-sphere representing such a cycle would be a holomorphic submanifold which turns out to be torsion in homology. Such a manifold can not be Kähler because in a Kähler manifold any complex submanifold is minimizing in it's volume class (it is calibrated by the Kähler form). Thus, the field theory and the topology agree. This also compares favourably with the results of [26], where a blowup of the conifold singularity produced a global geometry which was not of the Calabi-Yau type.

To see that such a sphere is torsion we go back to the orbifold with discrete torsion. There we have three complex lines of A_1 singularities meeting at the origin. On each of these lines one can assume that there is a blown-down $\mathbb{CP}^1$ fibration. However, we have a monodromy of the fractional branes [25], so we would think of this fibration as one where we change the orientation of the $\mathbb{CP}^1$ as we move on a circle about the origin. Such a $\mathbb{CP}^1$ can not be holomorphic because otherwise it would have a canonical orientation given by the complex structure.

This suggests that in the orbifolds with discrete torsion $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, which are deformed to a Calabi-Yau space with conifold singularities, the obstruction to the possibility of blowing up the conifold is a global feature of the topology and not a particular detail of the singularity itself. This seems to be in accordance with the results found in [26] and supports the idea that the local behavior near the singularity is universal.

To push this point of view further, let us consider the resolved conifold. It is known that the blow-up mode of the conifold is non normalizable in the conifold geometry.[4] Thus, given a metric deformation that changes the volume of this cycle, one can show that the metric perturbation is non normalizable. From this point of view, in order to determine if we are allowed to blow-up the conifold singularity we need to specify boundary conditions at infinity for this mode. As such, this mode is sensitive to the details of how the conifold is embedded in the Calabi-Yau space and can be expected to be dependent on the topological features at infinity.

[4] I thank J. Maldacena for showing me this result.

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4. Duality cascades, multi-trace operators and deformed moduli spaces.

In the conifold geometry we expect to be able to describe the duality cascades [12] in any UV description of the theory, and to generate as well a non-perturbative effective superpotential for brane probes.

The dualities are triggered from the effective gauge coupling becoming infinite for one of the fractional branes at a given scale μ. One would like a geometric interpretation of this same singularity in an associated system from where we can draw conclusion as to the nature of this duality.

The D-brane system under T-dualities is dual to a matrix model of D0-branes, where the gauge groups $\mathrm{U}(N) \times \mathrm{U}(M)$ count mutually BPS fractional branes. Since the model is associated with a deformation of an A_1 singularity, we will do the duality analysis in the matrix model for the ALE space instead, which is a considerable simplification.

Our point of view is to concentrate only on the role that the gauge coupling becoming infinite plays. Indeed, in the matrix model the two gauge couplings $1/g_1^2$ and $1/g_2^2$ are real parameters that are related to Wilson lines in M-theory, see for example [27]. The mass of a full D0-brane is proportional to $1/g_1^2 + 1/g_2^2$, which is a bound state of two fractional branes that are mutually BPS to the D0-brane if there are no F-terms or D-terms in the lagrangian, while the masses of the fractional branes are proportional to $1/g_i^2$.

The fractional branes have to be chosen so that each of them has positive D0-brane charge, however, when we move in the gauge coupling moduli space we pass through a point in M-theory where the Wilson line around the eleventh direction vanishes and one of the fractional D-branes becomes massless. This is exactly the point at infinite gauge coupling, but we are allowed to continue past this point as the moduli space of Wilson lines does not stop here.

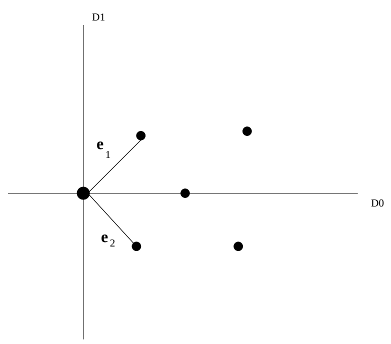

Figure 5: D-brane charge lattice for the A_1 singularity

The D-brane charge lattice has two charges, the D0-brane charge and the fractional D-brane charge. This is represented in the figure 5.

The way to choose the constituent fractional D-brane charges for the matrix model is to take those D-branes with the smallest possible *positive* D0-brane charge. When the Wilson line goes through zero and becomes negative we need to change variables to keep our prescription intact, and this corresponds to a change of basis in the D-brane lattice.

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The appropriate new lattice genertors are given by

$$e_2' = 2e_1 + e_2\,, \qquad e_1' = -e_1\,. \tag{4.1}$$

The lattice point $Ne_1 + Me_2$ in terms of the new variables is given by $(2M-N)e_1' + Me_2'$, and thus the new gauge group for the matrix model should be given by $\mathrm{U}(2M-N) \times \mathrm{U}(M)$, with the same quiver diagram.

We recognize this immediately as the Seiberg duality [22] transformation that appears in the duality cascade. As far as the enhanced SU(2) gauge model of M-theory, the change of basis $e_1 \to e_1' = -e_1$ on the root lattice of SU(2) is a Weyl reflection, a fact noticed in [28]. These changes by Weyl reflections are our choice as to which roots of the charge lattice are declared to be simple. In the matrix prescription mentioned above, this choice is given by which roots have the minimal positive D0-brane charge.

This toy model helps us understand why we have Seiberg Dualities in the field theory. However, notice that as far as M-theory is concerned we have to cross a singular point where we have extra massless fields, whereas in the conifold flow there is no such phenomena happening and the supergravity solution is non singular.

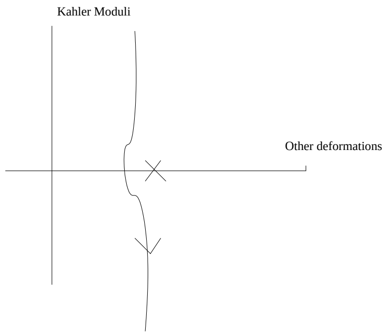

Figure 6: Deformed contour in closed string moduli space: the horizontal line represents the singularity in Kähler moduli space. The only real singularity is marked with the X.

The difference between the conifold and the A_1 singularity can be understood because the masses of these fractional branes also receive a contribution from the deformation of the A_1 singularity. We are now in four dimensions and not seven dimensions, so the A_1 enhanced symmetry point is broken by values of fields that depend on positions that are transverse to the four dimensions. However, the treatment of these moduli is not democratic. Some of them are determined by the complex structure of the Calabi-Yau manifold, while the gauge coupling comes from Kähler moduli. As far as the D-branes are concerned, the contributions to the central charge of a D-brane are determined by the Khäler moduli. Because of the extra deformation the would-be fractional D-branes are not BPS at this position; they are massive, but their local contribution to the central charge vanishes. This is depicted in figure 6, where we miss the singular point by taking a contour that takes into account that other fields than the gauge coupling are present.

We still want to keep a description where the local central charge of a fractional object contributes in the same direction as the global D3-brane charge. To ensure this, we have to change our notion of constituent fractional D-branes as we flow towards the infrared. As the contour in figure 6 shows, we have missed the singularity,

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but we still crossed the wall in Kähler moduli space, so we have an analytic continuation beyond the would-be singularity. This is also reminiscent of how brane crossing in D-brane setups give rise to Seiberg dualities [29]. The change in description in crossing the wall is the field theory duality. This cascade has been also generalized to SO / SP dualities by including orientifolds [30].

4.1 Additional evidence for the deformed conifold

In the paper of Klebanov and Strassler [12] it was proposed that at the end point of the duality cascades the singularity of the supergravity solution is resolved by a deformed conifold.

For this interpretation to make sense, one expects that this is the effective geometry that any number of D3-branes see. They proved that one probe D3-brane leads to a deformed conifold, and we want to move this result forward for a collection of n D3-brane probes.

One expects that this moduli space is essentially the symmetric product of n copies of the conifold geometry $\mathcal{M}$, namely one would want to find that the moduli space of vacua gives rise to $\mathrm{Sym}^n(\mathcal{M})$. We will prove this result in what follows.

Consider the theory for n D3-brane probes in the background of M fractional branes. Then we have a theory with gauge group $\mathrm{U}(n)\times\mathrm{U}(M+n)$, and the $\mathrm{U}(n+M)$ becomes strongly coupled with a dynamical scale Λ.

Remember the definition of the variables N_{ij} in eq. (1.4). The superpotential plus the non-perturbative correction from the Affleck-Dine-Seiberg superpotential [31] in these variables is given by

$$W = \mathrm{tr}\, N_{ij} N_{kl} \epsilon^{ik} \epsilon^{jl} + c \left(\frac{\Lambda^a}{\mathrm{Det}[N_{ij}]} \right)^b \tag{4.2}$$

with a, b, c constants depending on n, M and with

$$\det[N_{ij}] = \det \begin{pmatrix} N_{11} & N_{12} \\ N_{12} & N_{22} \end{pmatrix} \tag{4.3}$$

a $2n \times 2n$ matrix transforming as (2×2) matrices of $\mathrm{SU}(n)$ adjoints. The variation of the above equation leads to

$$\delta W = \mathrm{tr} \left[2N_{kl} \epsilon^{ik} \epsilon^{jl} \delta N_{ij} \right] + bc \left(\frac{\Lambda^a}{\mathrm{Det}[N_{ij}]} \right)^b \mathrm{tr} \left[\left(N^{-1} \right)^{ij} \delta N_{ij} \right] . \tag{4.4}$$

This is

$$\delta W = 2N_{kl} \epsilon^{ik} \epsilon^{jl} - bc \left(\frac{\Lambda^a}{\mathrm{Det}[N_{ij}]} \right)^b \left(N^{-1} \right)^{ij} \tag{4.5}$$

with N^{-1} the inverse of the $2n \times 2n$ matrix N_{ij}. Now notice that the term with the determinant has no $\mathrm{SU}(n)$ or flavor indices, so it can be treated as a constant and we can rewrite the equation we need to solve as

$$N_{kl} \epsilon^{ik} \epsilon^{jl} - \Omega \left(N^{-1} \right)^{ij} \tag{4.6}$$

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for some constant Ω. Multiplying by N on the right, we obtain

$$2N_{kl}\epsilon^{ik}\epsilon^{jl}N_{jl} = \Omega\delta^i_l \tag{4.7}$$

and a similar expression for matrix multiplication by N on the left. The content of the above equation is

$$\begin{aligned}\begin{pmatrix} N_{22} & -N_{21} \\ -N_{12} & N_{11} \end{pmatrix}\begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} &= \begin{pmatrix} \Omega & 0 \\ 0 & \Omega \end{pmatrix} \\ &= \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}\begin{pmatrix} N_{22} & -N_{21} \\ -N_{12} & N_{11} \end{pmatrix}.\end{aligned} \tag{4.8}$$

The above equations imply that N_{12} and N_{21} commute with N_{11} and N_{22}. If N_{11} is generic, we expect that it can be diagonalized with distinct eigenvalues by a complexified gauge transformation. Since N_{12} and N_{21} commute with N_{11}, it follows that they are diagonalized by the same basis. Then $[N_{12}, N_{21}] = [N_{11}, N_{22}] = 0$ so that all of the N_{ij} matrices commute with one another, and the collection of matrices N_{ij} in the adjoint of $\mathrm{SU}(n)$ are simultaneously diagonalizable.

The constraint satisfied by the eigenvalues is exactly the equation corresponding to the deformed conifold, so the moduli space of vacua is given by N copies of the deformed conifold geometry up to $\mathrm{SU}(N)$ gauge transformations which permute the eigenvalues. This is, the moduli space of vacua is the symmetric product of the conifold space geometry.

All we need to do now is match the scale Λ, or equivalently Ω, to the deformation parameter and we find exact agreement with the proposal of [12] to add an arbitrary number of probes to the geometry.

This is a strong test of the deformed conifold proposal, as the above result could have been expected to hold only in the semiclassical regime where the vevs of all of the N are very large and to receive large quantum deformations when the scale associated to the vevs $\langle N\rangle$ are small.

There are still singularities in the moduli space where two of these probe D3-branes meet. The low energy effective field theory degrees of freedom are such that one expects and enhanced $\mathrm{SU}(2)$ gauge theory. The field counting suggests that these have an effective $N = 4$ supersymmetry so that these singularities are not resolved.

4.2 More on discrete torsion

The statements made in the previous subsection depend only on effective field theory. However, one would want an ultraviolet description of the same statements in the theory with discrete torsion. Now we will try to do some consistency checks in this case to make sure that the geometrical description is accurate.

First, it is expected that we will deform the geometry without affecting the structure of the manifold at infinity. The constraints in eq. (2.2) are modified to quantum constraints

$$\gamma^2 = uvw - \lambda^2(u+v+w) + 2\lambda^3 + \epsilon \tag{4.9}$$

with ϵ being determined by the gaugino condensate. This becomes regular everywhere and it is the most economical possibility for the deformed quantum potential.

What is interesting in this case is that the constants a, b, c appearing in eq. (4.2) depend on n, M, the number of total branes and the number of fractional branes. In our case the number of fractional branes is counted by the superfield vev $\mathrm{tr}(A)$, and this is determined dynamically by the vacuum choice, as opposed to being imposed by hand as a superselection sector.

The superpotential might look like

$$W_{\mathrm{tree}} + a(\mathrm{tr}(A)) \left(\frac{\Lambda^b(\mathrm{tr}(A))}{\det(N)}\right)^c (\mathrm{tr}(A)) \tag{4.10}$$

which looks like a non-analytic function of the chiral fields and this might be considered pathological. However, the classical vev of $\mathrm{tr}(A)$ is quantized so these worries might be unfounded. The point is that, if we are trying to do a variation of the above equation, we need to worry about the vacuum structure even if there are no fractional branes because all the vacua of the theory with and without fractional branes are connected to one another. The only part which is difficult to trace is the variation with respect to A, but notice that in eq. (4.10) we can vary A assuming $\det(N)$ is constant. After all this is just one function on the moduli space of n-branes, and not a whole matrix variable, so we can treat is a constant matrix proportional to the identity. The same is true for values of $\mathrm{tr}(A)$ obtained after the variation of the superpotential. In the end, this variation might at worse renormalize the value of ζ (2.4), but one does not even expect this to happen unless $\mathrm{tr}(A) \neq 0$.

The non-perturbative superpontential should vanish for $\mathrm{tr}(A) = 0$ which is the condition for no fractional branes. This can be accomplished if there is a $\mathrm{tr}(A)^2$ prefactor in the non-perturbative terms of the superpotential. Since the theory is well defined in the UV with a given gauge coupling and value of ζ, the non-perturbative scale Λ can be written in terms of these UV quantities, and notice that $\zeta^{1/2} = \mathrm{tr}(A)$ for any of the two fractional brane solutions as well.

As such, these corrections will all be proportional to $\mathrm{tr}(A)$ which vanishes for any configuration without fractional branes, and we recover our undeformed conifold singularity when we expect it to be there.

Also, if $\mathrm{tr}(A) \neq 0$, then we get non-perturbative corrections to the geometry, and the low energy effective field theory near the conifold predicts that we obtain a deformed conifold.

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Notice that the non-perturbative form of the superpotential is of multi-trace type generically (the determinant can also be written in terms of multiple traces). Naïvely this seems to produce exotic non localities between D-branes [32], but in this case it seems that rather mysteriously we have obtained a very well defined local interpretation of the resulting geometry in terms of a deformed conifold.

5. Holography and universality

We want to argue that the phenomenon encountered previously for the conifold is generic: given a singularity in a Calabi-Yau manifold that is associated to a non-singular worldsheet $\mathcal{N} = (2,2)$ sigma model in type-II theories, without any RR gauge fields and without NS five-branes at the singularity, the classes of field theories associated to D-brane probes on the cone geometry are universal (up to Seiberg dualities). This is clear for singularities on a Calabi-Yau two-fold. They are all given by quotient singularities, and the resolutions of singularities by blow-ups are unique giving rise to an ADE Dynkin diagram for the exceptional divisors. This universal behavior has been found in toric singularities [33]–[36] and in the Weyl reflections for ADE fibrations in [28, 37]. The argument we present is speculative and should be considered more as a hint of universality rather than a proof. As such one should approach this section as establishing some evidence for a conjecture.

In order to describe this conjecture in more detail, we first need to give the data for a singularity. If a singularity $\mathcal{M}$ is isolated (there are no codimension two singularities that pass through the codimension three singularity), then the base of the cone over the singularity, which we will label as X is a smooth Sasakian manifold, see [4]. Let us also assume that X has no torsion homology.

If we put a set of D3-branes near the singularity then we will get a near horizon geometry of the form $AdS_5 \times X$. The moduli space of vacua of D3-branes should be given by the symmetric product $\mathrm{Sym}^n(\mathcal{M})$, and from the ideas of [38] one can actually measure this moduli space in terms of boundary conditions for the supergravity in $AdS_5 \times X$. From the geometry of X one can also determine the chiral ring of the associated large-N field theory completely, as well as the global symmetries of the field theory.

This data is universal regardless of what field theory realizes the singularity. Usually, given two field theories which have the same moduli space of vacua, the same global symmetries and the same chiral ring they can be dual to one another in the infrared [22].

If the singularity is non isolated we need to be more careful. This is because the supergravity approximation breaks down, as the singularities meet X on a circle. These codimension two singularities are going to be of ADE type on X. Thus, it is clear locally how the resolutions look like from analyzing the twisted sector strings. However, as these are on a circle, there is a possibility of having monodromies

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for the blow-up modes. These monodromies are given by the choices of discrete torsion on orbifolds [25], and there are consistency conditions that correlate the different monodromies on distinct circles. Presumably these consistency conditions are the ones that guarantee that the closed string field theory without the D-branes is non singular.

The monodromies are part of the data that can be measured away from the singularity, the moduli space of D-branes and the chiral ring are sensitive to these monodromies, so these are holomorphic observables as well. If we give the additional data of these monodromies, then we have specified the topological type of the singularity. Again, any two field theories with the same moduli space of vacua, chiral ring and global symmetries associated to the same singularity should be dual to one another.

The idea for our universality argument is that in the AdS/CFT correspondence the codimension three singularity disappears from the near horizon geometry and is encoded holographically on the base of the cone X at the boundary.

There are subtleties to this idea if X has torsion elements in $H^3(X, Z)$. In principle one can excite a topologically non-trivial H-field which is flat that would correspond to a choice of discrete torsion. This is holographic data, but at first sight it is hard to find holomorphic observables which correspond to these choices. Wrapping branes on these cycles gives us the spectrum of baryon-like operators (see [39]) in the conformal field theory and due to anomalies it is sensitive to this data. This additinal data is then necessary to establish the universality class of field theories at the singularity. In the conifold case, there is no torsion in homology, so there are no discrete torsion choices of this type to be made.

One would want a more constructive argument. Given a singularity, one would want to find a recipe for constructing a non-commutative resolution in the sense of [19], and then one has a well defined field theory associated to the singularity. On this field theory one can do various changes of D-brane basis which should correspond to Seiberg dualities, and these can be triggered by motion in the moduli space of couplings of the field theories, as in the duality cascades. One can expect that, in some sense, these dualities are triggered by crossing lines of marginal stability, and that they correspond to D-brane monodromies given by Fourier-Mukai transforms [40].

6. Conclusion

A careful analysis of the field theory associated to the conifold with discrete torsion has been presented. We saw that this field theory in the infrared is identical to the conifold field theory constructed by Klebanov and Witten [3]. As such, conifold singularities seem to have no additional discrete torsion degrees of freedom at the singularity. Obstructions to resolving the singularity are instead given by global topological considerations and depend on how the singularity is embedded in a particular Calabi-Yau space.

We have also presented some evidence that there are no additional discrete degrees of freedom at any Calabi-Yau singularity located at a point p that can not be guessed by data on the boundary of a finite size cone around p. This is a conjecture that states that the geometric classification of singularities depends on variables in the low energy string theory (like monodromies of fractional branes if certain cycles are of zero size) and the algebraic geometric type of the singularity.

The following properties are conjectured and are applicable so long as the Calabi-Yau manifold is considered to be very large (non compact). This is required because we want to guarantee that D3-branes are stable objects [41], only then we can trust that our field theory on the world volume of the D-branes is supersymmetric and that the heavy string modes can be integrated out.

1. Field theory duals of D-branes at Calabi-Yau singularities are uniquely determined by the singularity type up to Seiberg dualities [22]. The singularity type can include discrete torsion choices as either monodromies of fractional branes on codimension two singularities, or as torsion elements of $H^3(X,\mathbb{Z})$ if the singularity is a cone over X. Both of these data are holographic.

2. The singularities give rise to field theories that can have duality cascades if one can add fractional branes. The Seiberg dualities are changes of basis in the fractional brane lattice and correspond to crossing walls in Kähler moduli space.

3. One can also cross these walls by changing the boundary conditions at infinity (bare values of the couplings) and induce Seiberg dualities in theories without fractional branes.

4. Point-like branes in the Calabi-Yau manifold always fractionate at singularities, and never fractionate away from singularities.

One would also like to find a way of constructing field theories given a singularity. For toric singularities one has the proposal of [33], however, one would want a more general description for singularities which are non toric. As far as I know, all examples of Calabi-Yau threefold singularities can be obtained from either partial resolutions or deformations of known orbifold constructions; or by orbifolding known singularities. Perhaps one can address singularities systematically in this manner, as all of the field theories associated to these geometric processes would be constructible.

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References

[1] C. Vafa and E. Witten, *On orbifolds with discrete torsion*, *J. Geom. Phys.* **15** (1995) 189 [hep-th/9409188].

[2] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[3] I.R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, *Nucl. Phys.* **B 536** (1998) 199 [hep-th/9807080].

[4] D.R. Morrison and M.R. Plesser, *Non-spherical horizons, 1*, *Adv. Theor. Math. Phys.* **3** (1999) 1 [hep-th/9810201].

[5] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[6] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[7] M.R. Douglas, *D-branes and discrete torsion*, hep-th/9807235.

[8] M.R. Douglas and B. Fiol, *D-branes and discrete torsion, 2*, hep-th/9903031.

[9] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167.

[10] A. Strominger, *Massless black holes and conifolds in string theory*, *Nucl. Phys.* **B 451** (1995) 96 [hep-th/9504090].

[11] B.R. Greene, D.R. Morrison and A. Strominger, *Black hole condensation and the unification of string vacua*, *Nucl. Phys.* **B 451** (1995) 109 [hep-th/9504145].

[12] I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: duality cascades and χSB-resolution of naked singularities*, *J. High Energy Phys.* **08** (2000) 052 [hep-th/0007191].

[13] R. Gopakumar and C. Vafa, *Topological gravity as large-N topological gauge theory*, *Adv. Theor. Math. Phys.* **2** (1998) 413 [hep-th/9802016].

[14] F. Cachazo, K.A. Intriligator and C. Vafa, *A large-N duality via a geometric transition*, *Nucl. Phys.* **B 603** (2001) 3 [hep-th/0103067].

[15] K. Dasgupta, K. Oh and R. Tatar, *Geometric transition, large-N dualities and MQCD dynamics*, *Nucl. Phys.* **B 610** (2001) 331 [hep-th/0105066].

[16] K. Dasgupta, K. Oh and R. Tatar, *Open/closed string dualities and Seiberg duality from geometric transitions in M-theory*, hep-th/0106040.

[17] K. Dasgupta, K.-h. Oh, J. Park and R. Tatar, *Geometric transition versus cascading solution*, hep-th/0110050.

[18] D. Berenstein, V. Jejjala and R.G. Leigh, *Marginal and relevant deformations of $N = 4$ field theories and non-commutative moduli spaces of vacua*, *Nucl. Phys.* **B 589** (2000) 196 [hep-th/0005087].

[19] D. Berenstein and R.G. Leigh, *Resolution of stringy singularities by non-commutative algebras*, *J. High Energy Phys.* **06** (2001) 030 [hep-th/0105229].

[20] D. Berenstein and D. Grayson, *Strings on singularities and homological algebra*, work in progress.

[21] D. Berenstein and R.G. Leigh, *Non-commutative Calabi-Yau manifolds*, *Phys. Lett.* **B 499** (2001) 207 [hep-th/0009209].

[22] N. Seiberg, *Electric-magnetic duality in supersymmetric nonabelian gauge theories*, *Nucl. Phys.* **B 435** (1995) 129 [hep-th/9411149].

[23] R.G. Leigh and M.J. Strassler, *Exactly marginal operators and duality in four-dimensional $N = 1$ supersymmetric gauge theory*, *Nucl. Phys.* **B 447** (1995) 95 [hep-th/9503121].

[24] M.R. Douglas, B.R. Greene and D.R. Morrison, *Orbifold resolution by D-branes*, *Nucl. Phys.* **B 506** (1997) 84 [hep-th/9704151].

[25] D. Berenstein and R.G. Leigh, *Discrete torsion, AdS/CFT and duality*, *J. High Energy Phys.* **01** (2000) 038 [hep-th/0001055].

[26] P.S. Aspinwall, D.R. Morrison and M. Gross, *Stable singularities in string theory*, *Comm. Math. Phys.* **178** (1996) 115 [hep-th/9503208].

[27] D. Berenstein and R. Corrado, *Matrix theory on ALE spaces and wrapped membranes*, *Nucl. Phys.* **B 529** (1998) 225 [hep-th/9803048].

[28] F. Cachazo, S. Katz and C. Vafa, *Geometric transitions and $N = 1$ quiver theories*, hep-th/0108120.

[29] S. Elitzur, A. Giveon and D. Kutasov, *Branes and $N = 1$ duality in string theory*, *Phys. Lett.* **B 400** (1997) 269 [hep-th/9702014].

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[30] J.D. Edelstein, K. Oh and R. Tatar, *Orientifold, geometric transition and large-N duality for* SO / SP *gauge theories*, *J. High Energy Phys.* **05** (2001) 009 [hep-th/0104037].

[31] I. Affleck, M. Dine, and N. Seiberg, *Dynamical supersymmetry breaking in supersymmetric QCD*, *Nucl. Phys.* **B 241** (1984) 493.

[32] O. Aharony, M. Berkooz and E. Silverstein, *Multiple-trace operators and non-local string theories*, *J. High Energy Phys.* **08** (2001) 006 [hep-th/0105309].

[33] B. Feng, A. Hanany and Y.-H. He, *D-brane gauge theories from toric singularities and toric duality*, *Nucl. Phys.* **B 595** (2001) 165 [hep-th/0003085].

[34] B. Feng, A. Hanany and Y.-H. He, *Phase structure of D-brane gauge theories and toric duality*, *J. High Energy Phys.* **08** (2001) 040 [hep-th/0104259].

[35] C.E. Beasley and M.R. Plesser, *Toric duality is seiberg duality*, hep-th/0109053.

[36] B. Feng, A. Hanany, Y.-H. He and A.M. Uranga, *Toric duality as seiberg duality and brane diamonds*, hep-th/0109063.

[37] F. Cachazo, B. Fiol, K.A. Intriligator, S. Katz and C. Vafa, *A geometric unification of dualities*, hep-th/0110028.

[38] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, *Nucl. Phys.* **B 556** (1999) 89 [hep-th/9905104].

[39] E. Witten, *Baryons and branes in anti de Sitter space*, *J. High Energy Phys.* **07** (1998) 006 [hep-th/9805112].

[40] P.S. Aspinwall and M.R. Douglas, *D-brane stability and monodromy*, hep-th/0110071.

[41] P.S. Aspinwall and A.E. Lawrence, *Derived categories and zero-brane stability*, *J. High Energy Phys.* **08** (2001) 004 [hep-th/0104147].

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